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An extension of Mehta theorem with applications

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ABSTRACT

A remarkable fundamental theorem established by Mehta plays an important role in proving existence of fixed points, maximal elements, and equilibria in abstract economies. In this paper, we extend Himmelberg's measure of precompactness to the general setting of $l.c.$-spaces and obtain a generalization of Mehta's theorem. As an application, we develop some new fixed point theorems involving a kind of condensing mappings.

1. Introduction and preliminaries

In 1990, Mehta [7] established a key theorem about condensing maps in a Banach space by using the Kuratowski's measure of noncompactness. The result is very useful to prove the existence of fixed points for condensing maps. In 1993, Kim [5] generalized Mehta's result to a locally convex Hausdorff topological vector space by using the measure of precompactness due to Himmelberg et al. [2]. The purpose of this paper is to extend such a fundamental theorem to a general $l.c.$-space, and develop related propositions about projections and H-convexity in a product H-space.

We begin with some basic definitions and facts. For a nonempty set X, 2^X denotes the class of all subsets of X, and $\langle X\rangle$ denotes the class of all nonempty finite subsets of X. Recall that a pair $(X,\{\Gamma_A\})$ is called an H**-space**, if X is a topological space, together with a family $\{\Gamma_A\}$ of some nonempty contractible subsets of X indexed by $A\in\langle X\rangle$ such that $\Gamma_A\subseteq\Gamma_B$ whenever $A\subseteq B$. Given an H-space $(X,\{\Gamma_A\})$, a nonempty subset D of X is called to be H**-convex** if $\Gamma_A\subseteq D$ for all $A\subseteq\langle D\rangle$. For a nonempty subset K of X, we define the H**-convex hull** of K as

$$H\text{-co}K:=\bigcap\{D\mid D\text{ is }H\text{-convex in }X\text{ and }K\subseteq D\},$$

and we define the **closed** H**-convex hull** of K as

$$H\text{-}\overline{\text{co}}K:=\bigcap\{D\mid D\text{ is closed }H\text{-convex in }X\text{ and }K\subseteq D\}.$$

Notice that the intersection of H-convex sets is also an H-convex set if the intersection is nonempty. Therefore $H\text{-co}K$ and $H\text{-}\overline{\text{co}}K$ are the smallest H-convex set and closed H-convex set containing K, respectively. If K is a finite subset of X, then $H\text{-co}K$ is called a **polytope** in X. Further, $H\text{-co}K$ can be expressed as

$$H\text{-co}K=\bigcup\big\{H\text{-co}A\;\big|\;A\in\langle K\rangle\big\}.$$

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A **uniform structure** $\mathcal{U}$ for a set X is a nonempty family of subsets of $X \times X$ such that the following conditions hold:

(1) for any $U \in \mathcal{U}$, $(x,x) \in U$ for each $x \in X$,
(2) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$, where $U^{-1} := \{(x,y) \mid (y,x) \in U\}$,
(3) for any $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$, where

$$V \circ V := \{(x,y) \mid \text{there is a } z \in X \text{ such that } (x,z) \in V \text{ and } (z,x) \in V\},$$

(4) if $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$,
(5) if $U \subseteq V \subseteq X \times X$ and $U \in \mathcal{U}$, then $V \in \mathcal{U}$.

In this event, the pair $(X,\mathcal{U})$ is called a **uniform space**, whose topology induced by $\mathcal{U}$ is the family of all subsets G of X such that for each $x \in G$, there is a $V \in \mathcal{U}$ such that $V(x) \subseteq G$, where $V(x) := \{y \in X \mid (x,y) \in V\}$. Every member $V \in \mathcal{U}$ is called an **entourage**. An entourage V is **symmetric** provided that $(x,y) \in V$ implies $(y,x) \in V$. A subfamily $\mathcal{B}$ of a uniform structure $\mathcal{U}$ is called a **base** if each member of $\mathcal{U}$ contains a member of $\mathcal{B}$. In additions, for any subset K of X, its closure $\overline{K}$ can be expressed as $\overline{K} = \bigcap_{V \in \mathcal{B}} V(K)$. For details of uniform spaces, we refer to [1,4,10,12].

An *l.c.***-space** is an H-space $(X, \{\Gamma_A\})$ with a uniform structure $\mathcal{U}$ whose topology is induced by $\mathcal{U}$, and there is a base $\mathcal{B}$ consisting of symmetric entourages in $\mathcal{U}$ such that for each $V \in \mathcal{B}$, the set $V(E) := \bigcup_{x \in E} V(x)$ is H-convex whenever E is H-convex. We shall use the notation $(X,\mathcal{U},\mathcal{B})$ to stand for an *l.c.*-space. Equivalently, *l.c.*-spaces can be defined as a milder condition: $V(E)$ is H-convex whenever E is a polytope in X. Under this milder condition, we can prove that $V(E)$ is H-convex whenever E is H-convex. Indeed, for any finite set $A = \{x_1, x_2, \ldots, x_n\}$ of $V(E)$. By definition, there exists $B = \{y_1, y_2, \ldots, y_n\} \subseteq E$ such that $x_i \in V(y_i)$. It follows that

$$A = \{x_1, x_2, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} V(y_i) \subseteq V(B) \subseteq V(H\text{-}coB) \subseteq V(E).$$

Since $H\text{-}coB$ is a polytope in X, $V(H\text{-}coB)$ is H-convex and hence

$$\Gamma_A \subseteq V(H\text{-}coB) \subseteq V(E).$$

This shows that $V(E)$ is H-convex.

We note that for any $E \in \langle X \rangle$, $E \subseteq V(E)$ for all $V \in \mathcal{B}$. Furthermore, $V(E)$ can be expressed as

$$V(E) = \{x \in X \mid E \cap V(x) \neq \emptyset\}.$$

In fact, if $x \in V(E)$, then $x \in V(y)$ for some $y \in E$. That is, $(y,x) \in V$. Hence $(x,y) \in V$. It follows that $y \in V(x) \cap E \neq \emptyset$. Conversely, if $E \cap V(x) \neq \emptyset$, then there exists $y \in E$ with $y \in V(x)$. Now $(x,y) \in V$ implies $(y,x) \in V$ and hence $x \in V(y) \subseteq V(E)$.

In an *l.c.*-space $(X,\mathcal{U},\mathcal{B})$, a subset K of X is called **precompact**, if for any $V \in \mathcal{U}$, there exists a finite set F such that $K \subseteq V(F)$. An *l.c.*-space is called an *l.c.***-space with precompact polytope** if each polytope in X is precompact.

In an *l.c.*-space $(X,\mathcal{U},\mathcal{B})$, we define the **measure of precompactness** of a subset A in X by

$$Q(A) := \{V \in \mathcal{B} \mid A \subseteq \overline{V(K)} \text{ for some precompact set } K \text{ of } X\}.$$

Here, we notice that the measure $Q(A)$ can be rewritten as

$$Q(A) = \{V \in \mathcal{B} \mid A \subseteq \overline{V(K)} \text{ for some compact set } K \text{ of } X\}.$$

In fact, if $V \in Q(A)$, then $A \subseteq \overline{V(K)}$ for some precompact set K. This implies that $A \subseteq \overline{V(\overline{K})}$. Since K is precompact, $\overline{K}$ is compact; hence the result follows. Some related *l.c.*-spaces can be found in [3,4,8–11].

At first, we establish some basic properties, which will be used in our proofs.

Lemma 1.1. *If $(X,\mathcal{U},\mathcal{B})$ is an l.c.-space and K is H-convex in X, then $\overline{K}$ is also H-convex.*

Proof. Since $\mathcal{B}$ is a base, $\overline{K} = \bigcap_{V \in \mathcal{B}} V(K)$. Also, each $V(K)$ is H-convex for any $V \in \mathcal{B}$. Therefore, $\overline{K}$ is also H-convex. □

Lemma 1.2. *If $(X,\mathcal{U},\mathcal{B})$ is an l.c.-space and $K \subseteq X$, then $H\text{-}\overline{co}K = \overline{H\text{-}coK}$.*

Proof. By definition, $H\text{-}coK \subseteq H\text{-}\overline{co}K$. Further, since $H\text{-}\overline{co}K$ is closed, we have

$$\overline{H\text{-}coK} \subseteq \overline{H\text{-}\overline{co}K} = H\text{-}\overline{co}K.$$

On the other hand, $\overline{H\text{-}coK}$ is a closed H-convex set containing K, by Lemma 1.1. Since $H\text{-}\overline{co}K$ is the smallest closed H-convex set containing K, it follows that

$$H\text{-}\overline{co}K \subseteq \overline{H\text{-}coK}. \qquad \square$$

Lemma 1.3. *If $(X,\mathcal{U},\mathcal{B})$ is an l.c.-space with precompact polytope and K is precompact, then $H\text{-}coK$ is also precompact.*

Proof. For any $U\in\mathcal{U}$, we can choose a $V\in\mathcal{U}$ such that $V\circ V\subseteq U$. Since K is precompact, there exists a finite set F such that $K\subseteq V(F)$. Since $V(H\text{-}coF)$ is an H-convex set containing $V(F)$, we have

$$H\text{-}coK\subseteq H\text{-}coV(F)\subseteq V(H\text{-}coF).$$

By using the fact that the polytope $H\text{-}coF$ is precompact in X, we can find another finite set F' such that $H\text{-}coF\subseteq V(F')$. Therefore,

$$H\text{-}coK\subseteq V(H\text{-}coF)\subseteq V\big(V\big(F'\big)\big)=(V\circ V)\big(F'\big)\subseteq U\big(F'\big).$$

Since the above inclusion holds for arbitrary $U\in\mathcal{U}$, $H\text{-}coK$ is precompact. □

The following proposition is a very essential result w.r.t. the measure of precompactness.

Proposition 1.1. *Let $(X,\mathcal{U},\mathcal{B})$ be an l.c.-space with precompact polytope, and $A,B\subseteq X$. Then*

(1) *A is precompact iff $Q(A)=\mathcal{B}$,*
(2) *if $A\subseteq B$, then $Q(B)\subseteq Q(A)$,*
(3) *$Q(\overline{A})=Q(A)$,*
(4) *$Q(H\text{-}coA)=Q(A)$,*
(5) *$Q(A\cup B)=Q(A)\cap Q(B)$.*

Proof. For (1), suppose that A is a precompact set. Then for any $V\in B$, we have $(x,x)\in V$ for all $x\in A$. It follows that $x\in V(x)\subseteq V(A)$. This result implies that $A\subseteq V(A)\subseteq\overline{V(A)}$. This shows that $V\in Q(A)$ for all $V\in B$. Thus, $\mathcal{B}\subseteq Q(A)$ and hence $\mathcal{B}=Q(A)$. Conversely, suppose $Q(A)=\mathcal{B}$. Then for any $U\in\mathcal{U}$, there exists a $V\in\mathcal{U}$ such that $V\circ V\subseteq U$. Since $\mathcal{B}$ is a base, we can take this $V\in\mathcal{B}$. Similarly, we have some $V'\in\mathcal{B}$ such that $V'\circ V'\subseteq V$. Since $V'\in\mathcal{B}=Q(A)$, there exists a precompact set K such that $A\subseteq\overline{V'(K)}$. Further, we have a finite set F such that $K\subseteq V'(F)$. Thus,

$$A\subseteq\overline{V'(K)}\subseteq\overline{V'\big(V'(F)\big)}=\overline{\big(V'\circ V'\big)(F)}\subseteq\overline{V(F)}\subseteq V\big(V(F)\big)=(V\circ V)(F)\subseteq U(F).$$

Since $U\in\mathcal{U}$ is arbitrary, A is precompact.

For (2), if $A\subseteq B$, then for any $V\in Q(B)$, $B\subseteq\overline{V(K)}$ for some precompact set K. So $A\subseteq B\subseteq\overline{V(K)}$, and hence $V\in Q(A)$. This means that $Q(B)\subseteq Q(A)$.

For (3), it is sufficient to prove that $Q(A)\subseteq Q(\overline{A})$. Indeed, if $V\in Q(A)$, then there is a precompact set K such that $A\subseteq\overline{V(K)}$. It follows that $\overline{A}\subseteq\overline{V(K)}$ and hence $V\in Q(\overline{A})$.

For (4), we have to show that $Q(A)\subseteq Q(H\text{-}coA)$. If $V\in Q(A)$, then $A\subseteq\overline{V(K)}$ for some precompact set K. Consequently,

$$H\text{-}coA\subseteq H\text{-}co\overline{V(K)}\subseteq\overline{V(H\text{-}coK)},$$

since $\overline{V(H\text{-}coK)}$ is an H-convex set containing $\overline{V(K)}$ by Lemma 1.1. Since $H\text{-}coK$ is precompact by Lemma 1.3, it follows that $V\in Q(H\text{-}coA)$. This shows that $Q(A)\subseteq Q(H\text{-}coA)$.

For (5), since $Q(A\cup B)\subseteq Q(A)$ and $Q(A\cup B)\subseteq Q(B)$, we have $Q(A\cup B)\subseteq Q(A)\cap Q(B)$. Conversely, suppose $V\in Q(A)\cap Q(B)$. Then there exist precompact sets K_1 and K_2 such that $A\subseteq\overline{V(K_1)}$ and $B\subseteq\overline{V(K_2)}$. Hence

$$A\cup B\subseteq\overline{V(K_1)}\cup\overline{V(K_2)}\subseteq\overline{V(K_1\cup K_2)}.$$

Since $K_1\cup K_2$ is also precompact, it follows that $V\in Q(A\cup B)$, and hence $Q(A)\cap Q(B)\subseteq Q(A\cup B)$. □

Corollary 1.1. *If $(X,\mathcal{U},\mathcal{B})$ is an l.c.-space with precompact polytope, then*

$$Q(A)=Q(\overline{H\text{-}coA})=Q(H\text{-}\overline{co}A),\quad\forall A\subseteq X.$$

Besides, we review some concepts about the product of H-spaces. Let $\{(X_\alpha,\{\Gamma^\alpha_{A_\alpha}\})\mid\alpha\in I\}$ be a family of H-spaces, where I is a finite or infinite index set. Define $X:=\prod_{\alpha\in I}X_\alpha$ to be the product space with product topology, and for each $\alpha\in I$, let $P_\alpha:X\to X_\alpha$ denote the projection of X onto X_α. For any finite set A of X, we set $\Gamma_A:=\prod_{\alpha\in I}\Gamma^\alpha_{A_\alpha}$, where $A_\alpha=P_\alpha(A)$ for each $\alpha\in I$. Then $(X,\{\Gamma_A\})$ forms an H-space.

In fact, since for each $\alpha\in I$, $\Gamma^\alpha_{A_\alpha}$ is contractible, it is easy to see that Γ_A is contractible. Moreover, if A and B are two finite subsets of X with $A\subseteq B$, then for each $\alpha\in I$, $P_\alpha(A)\subseteq P_\alpha(B)$; that is, $A_\alpha\subseteq B_\alpha$ and hence $\Gamma^\alpha_{A_\alpha}\subseteq\Gamma^\alpha_{B_\alpha}$. Thus,

$$\Gamma_A=\prod_{\alpha\in I}\Gamma^\alpha_{A_\alpha}\subseteq\prod_{\alpha\in I}\Gamma^\alpha_{B_\alpha}=\Gamma_B.$$

Therefore, $(X,\{\Gamma_A\})$ is an H-space. Under this terminology, we have the following lemma due to Tarafdar [9].

Lemma 1.4. *The product $X := \prod_{\alpha\in I} X_\alpha$ of any number (finite or infinite) of H-spaces X_α ($\alpha \in I$) is an H-space, and the product of H-convex sets is H-convex.*

Lemma 1.5. *The projection of an H-convex set in the product H-space $X := \prod_{\alpha\in I} X_\alpha$ is also H-convex.*

Proof. Suppose K is an H-convex subset of $X = \prod_{\alpha\in I} X_\alpha$ and K_α is the projection of K onto X_α. We want to show that K_α is an H-convex subset of X_α. Let A_α be a nonempty finite subset of K_α and $P_\alpha : X \to 2^{X_\alpha}$ denote the projection of X onto X_α. Then there is a correspondent nonempty finite subset A of K such that $P_\alpha(A) = A_\alpha$. Since K is H-convex, we have $\Gamma_A \subseteq K$. It follows from the definition of Γ_A that

$$\Gamma^\alpha_{A_\alpha} = P_\alpha(\Gamma_A) \subseteq P_\alpha(K) = K_\alpha.$$

Thus, K_α is H-convex. □

Lemma 1.6. *If $P_\alpha : X \to 2^{X_\alpha}$ is the projection of the product H-space $X := \prod_{\alpha\in I} X_\alpha$ onto X_α, and K is a nonempty subset of X, then*

$$P_\alpha(H\text{-}\overline{co}K) \subseteq H\text{-}\overline{co}P_\alpha(K).$$

Proof. If $y \in P_\alpha(H\text{-}\overline{co}K)$, then $y = P_\alpha(x)$ for some $x \in H\text{-}\overline{co}K$. Thus, $y = P_\alpha(x) \in P_\alpha(D)$ for all closed H-convex subsets D containing K. Now for any closed H-convex set D_α containing $P_\alpha(K)$, we define

$$D = D_\alpha \times \left(\prod_{\beta\neq\alpha} X_\beta\right).$$

Then D is a closed H-convex set containing K, and $D_\alpha = P_\alpha(D)$. Therefore, $y \in P_\alpha(D) = D_\alpha$ for all closed H-convex set D_α containing $P_\alpha(K)$. It follows that

$$y \in \bigcap\bigl\{D_\alpha \bigm| D_\alpha \text{ is closed } H\text{-convex and } P_\alpha(K) \subseteq D_\alpha\bigr\} = H\text{-}\overline{co}P_\alpha(K).$$

Thus, $P_\alpha(H\text{-}\overline{co}K) \subseteq H\text{-}\overline{co}P_\alpha(K)$. □

2. A fundamental theorem

Let $\{(X_\alpha, \mathcal{U}_\alpha, \mathcal{B}_\alpha) \mid \alpha \in I\}$ be a family of $l.c.$-spaces with precompact polytope, where I is a finite or infinite index set and let $X := \prod_{\alpha\in I} X_\alpha$ be the product H-space. For each $\alpha \in I$, let Q_α be a measure of precompactness in X_α. We shall say that a set-valued mapping $T_\alpha : X \to 2^{X_\alpha}$ is Q_α**-condensing**, provided that $Q_\alpha(P_\alpha(A)) \subsetneq Q_\alpha(T_\alpha(A))$ for every A satisfying $P_\alpha(A)$ is a nonprecompact subset of X_α. It is easy to check that $T_\alpha : X \to 2^{X_\alpha}$ is Q_α-condensing, whenever X is compact. Also, under the particular case where $I = \{1\}$, the projection P_α is just the identity on X; therefore, the above definition reduces to the usual Q-condensing mapping $T : X \to 2^X$; that is, $Q(A) \subsetneq Q(T(A))$ for every nonprecompact set $A \subseteq X$. For details, see for example [2,3,5,6].

We are ready to prove our fundamental theorem.

Theorem 2.1. *Let $\{(X_\alpha, \{\Gamma^\alpha_{A_\alpha}\}) \mid \alpha \in I\}$ be a family of l.c.-spaces with precompact polytope, and let $X := \prod_{\alpha\in I} X_\alpha$. Suppose $T_\alpha : X \to 2^{X_\alpha}$ is a Q_α-condensing mapping for each $\alpha \in I$. Then there exists a nonempty compact H-convex subset K of X, with $K = \prod_{\alpha\in I} K_\alpha$, such that $T_\alpha(K) \subseteq K_\alpha$ for each $\alpha \in I$; that is, $T_\alpha : K \to 2^{K_\alpha}$ for each $\alpha \in I$.*

Proof. Fix any $x_0 \in X = \prod_{\alpha\in I} X_\alpha$. Let $\mathcal{F}$ be the family of all closed H-convex subsets C of X which contains x_0 and satisfies the following conditions: $C = \prod_{\alpha\in I} C_\alpha$, where C_α are closed H-convex subsets of X_α such that $T_\alpha(C) \subseteq C_\alpha$ for each $\alpha \in I$, and let $\mathcal{F}_\alpha$ be the family of such sets C_α. Now let $K = \bigcap_{C\in\mathcal{F}} C$ and $K_\alpha = \bigcap_{C_\alpha\in\mathcal{F}_\alpha} C_\alpha$ for each $\alpha \in I$. Then $x_0 \in K$ and $P_\alpha(x_0) \in K_\alpha$. Hence K and K_α are nonempty closed H-convex sets. Furthermore, we can show that $K = \prod_{\alpha\in I} K_\alpha$. In fact, if $x \in K$, then $x \in C$ for all $C \in \mathcal{F}$. By the definition of $\mathcal{F}$, we have $P_\alpha(x) \in C_\alpha$ for all $C_\alpha \in \mathcal{F}_\alpha$. It follows that

$$P_\alpha(x) \in \bigcap_{C_\alpha\in\mathcal{F}_\alpha} C_\alpha = K_\alpha \quad \text{for each } \alpha \in I.$$

Thus, $x \in \prod_{\alpha\in I} K_\alpha$. Conversely, if $x \in \prod_{\alpha\in I} K_\alpha$, then $P_\alpha(x) \in K_\alpha$ for each $\alpha \in I$. Thus, $P_\alpha(x) \in C_\alpha$ for all $C_\alpha \in \mathcal{F}_\alpha$ for each $\alpha \in I$, and hence $x \in C$ for all $C \in \mathcal{F}$. This implies that $x \in K$. Next, we show that $T_\alpha(K) \subseteq K_\alpha$ for each $\alpha \in I$. For any $x \in K$, we have $x \in C$ for all $C \in \mathcal{F}$. By the definition of $\mathcal{F}$, $T_\alpha(x) \in C_\alpha$ for all $C_\alpha \in \mathcal{F}_\alpha$. Hence $T_\alpha(x) \in K_\alpha$. That is, $T_\alpha(K) \subseteq K_\alpha$ for each $\alpha \in I$.

It remains to show that K is compact. To deal with this, we define $T : X \to 2^X$ by $T(x) := \prod_{\alpha\in I} T_\alpha(x)$, and let

$$K' := H\text{-}\overline{co}\bigl(\{x_0\} \cup T(K)\bigr).$$

Then $K' \subseteq K$, since K is a closed H-convex set containing both x_0 and $T(K)$. For each $\alpha \in I$, let $K'_\alpha = P_\alpha(K')$. Then

$$K'_\alpha = P_\alpha\big(K'\big) \subseteq P_\alpha(K) = K_\alpha.$$

Applying Lemma 1.1, together with Lemma 1.4 and Lemma 1.5, we can obtain a closed H-convex set K'', defined by

$$K'' := \prod_{\alpha \in I} \overline{K'_\alpha}.$$

Also, $x_0 \in K'$ implies that $P_\alpha(x_0) \in K'_\alpha \subseteq \overline{K'_\alpha}$ for each $\alpha \in I$, and hence

$$x_0 \in \prod_{\alpha \in I} P_\alpha(x_0) \subseteq \prod_{\alpha \in I} \overline{K'_\alpha} = K''.$$

Clearly, we have

$$K'' = \prod_{\alpha \in I} \overline{K'_\alpha} \subseteq \prod_{\alpha \in I} \overline{K_\alpha} = \prod_{\alpha \in I} K_\alpha = K.$$

On the other hand, for any $x \in K''$,

$$T(x) \subseteq T\big(K''\big) \subseteq T(K) \subseteq K'.$$

Hence

$$T_\alpha(x) = P_\alpha\big(T(x)\big) \subseteq P_\alpha\big(K'\big) = K'_\alpha \subseteq \overline{K'_\alpha}.$$

That is, $T_\alpha(K'') \subseteq \overline{K'_\alpha}$. Thus, $K'' \in \mathcal{F}$ and hence $K \subseteq K''$. So we conclude that $K = K''$ and hence

$$K_\alpha = P_\alpha(K) = P_\alpha\big(K''\big) = \overline{K'_\alpha}.$$

Finally, by Lemma 1.5, we can easily check that

$$\begin{aligned}
\overline{K'_\alpha} &= \overline{P_\alpha\big(K'\big)} = \overline{P_\alpha\big(H\text{-}\overline{co}\{x_0\} \cup T(K)\big)} \\
&\subseteq \overline{H\text{-}\overline{co}P_\alpha\big(\{x_0\} \cup T(K)\big)} \\
&= H\text{-}\overline{co}\big(P_\alpha\big(\{x_0\} \cup T(K)\big)\big) \\
&= H\text{-}\overline{co}\big(P_\alpha\big(\{x_0\}\big) \cup P_\alpha\big(T(K)\big)\big) \\
&= H\text{-}\overline{co}\big(P_\alpha\big(\{x_0\}\big) \cup T_\alpha(K)\big).
\end{aligned} \tag{1}$$

To show that K is compact, it is sufficient to show that each K_α is compact by Tychonoff theorem. Assume that K_α is not compact for some $\alpha \in I$. Thus, K_α is not precompact since K_α is closed. It follows that

$$Q_\alpha(K_\alpha) = Q_\alpha\big(P_\alpha(K)\big) \subsetneq Q_\alpha\big(T_\alpha(K)\big).$$

Applying Proposition 1.1 and Corollary 1.1, we have

$$\begin{aligned}
Q_\alpha\big(\overline{K'_\alpha}\big) &\supseteq Q_\alpha\big(H\text{-}\overline{co}\big(P_\alpha\big(\{x_0\}\big) \cup T_\alpha(K)\big)\big) \\
&= Q_\alpha\big(P_\alpha\big(\{x_0\}\big) \cup T_\alpha(K)\big) \\
&= Q_\alpha\big(P_\alpha(x_0)\big) \cap Q_\alpha\big(T_\alpha(K)\big) \\
&= \mathcal{B} \cap Q_\alpha\big(T_\alpha(K)\big) \\
&= Q_\alpha\big(T_\alpha(K)\big) \supsetneq Q_\alpha(K_\alpha),
\end{aligned} \tag{2}$$

which contradicts with the fact $K_\alpha = \overline{K'_\alpha}$. Therefore, K_α is compact for each $\alpha \in I$ and the proof is complete. □

In particular, when $I = \{1\}$, we have the following:

Corollary 2.1. *Let $(X, \mathcal{U}, \mathcal{B})$ be an l.c.-space with precompact polytope. If $T : X \to 2^X$ is a Q-condensing mapping, then there exists a nonempty compact H-convex subset K of X such that $T(K) \subseteq K$.*

We remark that Corollary 2.1 generalizes Mehta's result in Banach spaces [7] and Kim's result in locally convex topological vector spaces [5]. A precompact version in locally G-convex spaces can be found in [3], where the derived set K is precompact, instead of a compact set.

3. Applications to fixed point theorems

In 1992, Tarafdar [8] proved the following fixed point theorem:

Theorem A. *Let $X = \prod_{\alpha\in I} X_\alpha$ be the product space of compact H-spaces X_α, $\alpha \in I$. Suppose that $T_\alpha : X \to 2^{X_\alpha}$ satisfies the following conditions for each $\alpha \in I$.*

(1) *For each $x \in X$, $T_\alpha(x)$ is a nonempty H-convex subset of X_α for each $\alpha \in I$.*
(2) *For each $x_\alpha \in X_\alpha$, $T_\alpha^{-1}(x_\alpha)$ contains an open subset O_{x_α} of X such that $\bigcup_{x_\alpha\in X_\alpha} O_{x_\alpha} = X$ (where O_{x_α} may be empty for some x_α).*

Then $T := \prod_{\alpha\in I} T_\alpha$ has a fixed point.

Recall that a set-valued mapping $T : X \to 2^X$ is **upper semicontinuous** ($u.s.c.$), if for each $x \in X$ and for any open set G containing $T(x)$, there is an open neighborhood U of x such that $T(y) \subseteq G$ for all $y \in U$. In 1997, Tarafdar and Watson [10] established the following fixed point theorem for upper semicontinuous set-valued mappings in a compact $l.c.$-space.

Theorem B. *Let $(X, \mathcal{U}, \mathcal{B})$ be a compact l.c.-space. If $T : X \to 2^X$ is an upper semicontinuous set-valued mapping with compact H-convex values, then T has a fixed point.*

We note that a nonempty subset D of a topological space X is said to be **compactly open**, if $D \cap K$ is open for all compact subsets K of X. Based on the above results, we are able to show two generalized fixed point theorems as follows.

Theorem 3.1. *Let $X := \prod_{\alpha\in I} X_\alpha$ be the product space of $l.c.$-spaces $(X_\alpha, \mathcal{U}_\alpha, \mathcal{B}_\alpha)$, $\alpha \in I$ with precompact polytope and let $T_\alpha : X \to 2^{X_\alpha}$ be a Q_α-condensing mapping for each $\alpha \in I$. Suppose that T_α satisfies the following conditions for each $\alpha \in I$.*

(1) *For each $x \in X$, $T_\alpha(x)$ is a nonempty H-convex subset of X_α for each $\alpha \in I$.*
(2) *For each $x_\alpha \in X_\alpha$, $T_\alpha^{-1}(x_\alpha)$ contains a compactly open subset O_{x_α} of X such that $\bigcup_{x_\alpha\in X_\alpha} O_{x_\alpha} = X$ (where O_{x_α} may be empty for some x_α).*

Then $T := \prod_{\alpha\in I} T_\alpha$ has a fixed point; that is, there exists $x = (x_\alpha)$ such that $x_\alpha \in T_\alpha(x)$ for each α.

Proof. By Theorem 2.1, there exists a nonempty compact H-convex subset K of X with $K = \prod_{\alpha\in I} K_\alpha$ and $T_\alpha(K) \subseteq K_\alpha$. Also, for each $x \in K \subseteq X$, each $T_\alpha(x)$ is a nonempty H-convex subset of K_α by (1). Further, by (2), we have some $x_\alpha \in X_\alpha$ such that

$$x \in O_{x_\alpha} \subseteq T_\alpha^{-1}(x_\alpha).$$

Equivalently,

$$x_\alpha \in T_\alpha(x) \subseteq T_\alpha(K) \subseteq K_\alpha.$$

This yields $K \subseteq \bigcup_{x_\alpha\in K_\alpha} O_{x_\alpha}$. It follows that

$$K = \left(\bigcup_{x_\alpha\in K_\alpha} O_{x_\alpha}\right) \cap K = \bigcup_{x_\alpha\in K_\alpha} (O_{x_\alpha} \cap K).$$

Define

$$O'_{x_\alpha} := O_{x_\alpha} \cap K \quad \text{for each } x_\alpha \in K_\alpha.$$

Then each O'_{x_α} is an open subset of K satisfying $K = \bigcup_{x_\alpha\in K_\alpha} O'_{x_\alpha}$, and $O'_{x_\alpha} \subseteq O_{x_\alpha} \subseteq T^{-1}(x_\alpha)$ for each $x_\alpha \in K_\alpha$. All conditions of Theorem A are fulfilled w.r.t. $T_\alpha : K \longrightarrow 2^{K_\alpha}$, and hence T has a fixed point. □

As a consequence, we obtain the following corollary:

Corollary 3.1. *Let $(X, \mathcal{U}, \mathcal{B})$ be an $l.c.$-space with precompact polytope. Suppose that $T : X \to 2^X$ is a Q-condensing mapping with nonempty H-convex values such that for each $y \in X$, $T^{-1}(y)$ contains a compactly open subset O_y of X with $\bigcup_{y\in X} O_y = X$. Then T has a fixed point.*

Theorem 3.2. *Let $X := \prod_{\alpha\in I} X_\alpha$ be the product space of $l.c.$-spaces $(X_\alpha, \mathcal{U}_\alpha, \mathcal{B}_\alpha)$ with precompact polytope, where $\alpha \in I$. If each $T_\alpha : X \to 2^{X_\alpha}$ is an upper semicontinuous Q_α-condensing mapping with nonempty H-convex values, then $T := \prod_{\alpha\in I} T_\alpha$ has a fixed point.*

Proof. By Theorem 2.1, there exists a nonempty compact H-convex subset K of X with $K = \prod_{\alpha \in I} K_\alpha$ and $T_\alpha(K) \subseteq K_\alpha$ for each $\alpha \in I$. Clearly, the restriction $T_\alpha : K \to K_\alpha$ is also $u.s.c.$, and each $T_\alpha(x)$ is nonempty, compact and H-convex for all $x \in K$. Since $T_\alpha(K) \subseteq K_\alpha$, we obtain

$$T(x) = \prod_{\alpha \in I} T_\alpha(x) \subseteq \prod_{\alpha \in I} K_\alpha = K, \quad \forall x \in K.$$

That is, $T(K) \subseteq K$. Applying Lemma 1.1, we note that T is also $u.s.c.$ with nonempty H-convex values. Thus, it follows from Theorem B that T has a fixed point. □

In case $I = \{1\}$, we have the following immediate result.

Corollary 3.2. *Let $(X, \mathcal{U}, \mathcal{B})$ be an l.c.-space with precompact polytope. Suppose that $T : X \to 2^X$ is an upper semicontinuous Q-condensing mapping with H-convex values. Then T has a fixed point.*

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